# LINEAR PARAMETRIZATION OF DEMANDS AND SUPPLIES IN TRANSPORTATION PROBLEMS

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## Introduction

Transportation problems (TP) represent a well known and frequently used part of linear programming (LP). As compared with general LP problems, the TP possess some specific features which enabled to develop specific methods of their solution. Due to accelerated expansion of information technologies, usually little attention is paid to TP with parametres; the capacity and efficiency of the contemporaneous personal computers make it easy, to compute TP with new input data from the very beginning.

Parametric problems allow to observe relations between the optimal solution of the model and the modification of input data. Acceptable modification of the data are measured through variable parametres, conditioning the optimum solution.

Purpose of this paper is to show some possibilities presented by the solution of a simple parametric TP to be taken into account, while evaluating the transportation system and deciding about the transportation programme.

## Linear Parametrization of the Supply and Demand Vector

Let us suppose a simple TP with parametres

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \quad (\min) \tag{1}$$

$$\sum_{j=1}^{n} x_{ij} = a_i + \lambda^{(a_i)}, \; i = 1, ..., m \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} = b_j + \lambda^{(b_j)}, \; j = 1, ..., m \tag{3}$$

$$x_{ij} \geq 0, \; i = 1, ..., m, \; j = 1, ..., n \tag{4}$$

$$\text{and} \sum_{j=1}^{n} a_i + \lambda^{(a_i)} = \sum_{i=1}^{m} b_j + \lambda^{(b_j)} \tag{5}$$

where: $\lambda^{(a_i)}$ – scallar parameter expressing modification of the $i$-th supply capacity $a_i$
$\lambda^{(b_j)}$ – scallar parameter expressing modification of the $j$-th demand $b_j$

Let us indicate the supply-demand vector $\mathbf{b} = (a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_{n-1})^T$ and the matrix of basis **B**, which is a sub-matrix of the whole system of coefficients $\mathbf{A} = (a_{ij})$; the coefficient vectors of variables $x_{ij}$ have the specific form, i.e.

$$\mathbf{a}_{ij} = \begin{pmatrix} \mathbf{e}_i \\ \mathbf{e}_j \end{pmatrix}$$

where: $\mathbf{e}_i = (0, \ldots, 1, 0, \ldots 0)^T \quad i = 1, 2, \ldots, m;\ e_{ii} = 1$, i.e. $e_{ik} = 0,\ k \neq i$
$\mathbf{e}_j = (0, \ldots, 1, 0, \ldots 0)^T \quad j = 1, 2, \ldots, n;\ e_{jj} = 1$, i.e. $e_{jk} = 0,\ k \neq j$

Matrix $\mathbf{B}^{-1} = (\sigma_{ij})$, $\sigma_{ij} \in \{-1, 0, 1\}$ is the transformation matrix. Solution of the problem (1) to (5), supposing that just the $k$-th coefficient of the supply-demand vector **b** $(1 \leq k \leq m + n - 1)$ is modifying by $\lambda_k$, which is defined by the expression

$$\mathbf{x}_{B\lambda} = \mathbf{B}^{-1}(\mathbf{b} + \lambda_k \mathbf{e}_k) \tag{6}$$

which equals to

$$\mathbf{x}_{B\lambda} = \mathbf{B}^{-1}\mathbf{b} + \lambda_k \mathbf{B}^{-1}\mathbf{e}_k \tag{7}$$

and further to

$$x_{B\lambda} = x_{ij} + \lambda_k A_{ik}\ (i, j) \in N \tag{8}$$

where $\mathbf{A}_{ik}$ are coefficients of the $k$-th column in matrix $\mathbf{B}^{-1}$ and $N_1$ is the set of indices related to basic variables. To keep the solution feasible in the same basis the following condition has to be satisfied:

$$x_{ij} + \lambda_k A_{ik} \geq 0,\ (i, j) \in N_1 \tag{9}$$

The set of inequations (9) defines the interval of feasible values of parameter $\lambda_k$. Let's analyze it! The $A_{ik}$ elements of matrix $\mathbf{B}^{-1}$ can acquire the values 1, –1, 0. Therefore, the solution of the set (9) for $\mathbf{A}_{ik} = 1$ will become the inequation

$$\lambda_k \leq \max_{i}\left(-\frac{x_{ij}}{\mathbf{A}_{ik}}\right) = \underline{\theta}^{(k)} \quad (i, j) \in N_1 \tag{10}$$

and for $\mathbf{A}_{ik} = -1$



$$\lambda_k \leq \min_{i}\left(-\frac{x_{ij}}{\mathbf{A}_{ik}}\right) = \overline{\theta}^{(k)}\ (i, j) \in N_1 \tag{11}$$

Intersection of both these intervals (10) and (11) defines then the interval

$$\underline{\theta}^{(k)} \leq \lambda_k \leq \overline{\theta}^{(k)} \tag{12}$$

If $\mathbf{A}_{ik} = 1$ does not exist, we suppose $\underline{\theta}^{(k)} = -\infty$;
If $\mathbf{A}_{ik} = -1$ does not exist, we suppose $\overline{\theta}^{(k)} = \infty$.

In case of simultaneous modification of several or all the supply-demand items of a TP by parametres $\lambda^{(ai)}$, respectively $\lambda^{(bj)}$ and, assuming $B^{-1}$ being the transformation matrix, the solution of the problem can be defined by means of the following relation:

$$\mathbf{x}_{B\lambda} = \mathbf{B}^{-1}(\mathbf{b} + \lambda) \tag{13}$$

where $\mathbf{x} = (\lambda^{a1}, \ldots, \lambda^{am}, \lambda^{b1}, \ldots, \lambda^{(b_n - 1)})^T$

from which it follows

$$\mathbf{x}_{B\lambda} = \mathbf{x}_B + \mathbf{B}^{-1}\lambda \tag{14}$$

If we wish to keep the solution feasible in the optimal basis **B**, it is needed

$$\mathbf{x}_B + \mathbf{B}^{-1}\lambda \geq 0 \tag{15}$$

$$\text{respectively} \quad \mathbf{B}^{-1}\lambda \geq -\mathbf{x}_B \tag{16}$$

The set of inequations (16) defines the area of feasible values of parametres.

**Application**

To illustrate the above, a small numerical example is shown below. In Tab. I the optimal solution of a TP is given.

I. The optimal solution of a TP

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $a_i$ |
|---|---|---|---|---|---|
| $D_1$ | 12 | 12 | 100 8 | 13 | 100 |
| $D_2$ | 80 7 | 10 | 20 7 | 150 11 | 250 |
| $D_3$ | 70 6 | 130 9 | 12 | 11 | 200 |
| $b_j$ | 150 | 130 | 120 | 150 | 550 |

According to (10) and (11) the intervals of feasible parameter values are as follows:

$$-100 \le \lambda^{(a_1)} \le 20$$
$$-150 \le \lambda^{(a_2)} \le \infty$$
$$-70 \le \lambda^{(a_3)} \le 80$$
$$-80 \le \lambda^{(b_1)} \le 150$$
$$-80 \le \lambda^{(b_2)} \le 70$$
$$-20 \le \lambda^{(b_3)} \le 150$$

and intervals for feasible modification of the supply-demand values, are

$a_1 \in \langle 0;\ 120 \rangle$
$a_2 \in \langle 100;\ \infty \rangle$
$a_3 \in \langle 130;\ 280 \rangle$
$b_1 \in \langle 70;\ 300 \rangle$
$b_2 \in \langle 50;\ 200 \rangle$
$b_3 \in \langle 100;\ 270 \rangle$

According to (16), the area of feasible parameter values is defined by the set of following inequations:

$$\begin{array}{rrrrrrl}
\lambda^{(a_1)} & & & & & & \ge -100 \\
 & & -\lambda^{(a_3)} & +\lambda^{(b_1)} & +\lambda^{(b_2)} & & \ge -80 \\
-\lambda^{(a_1)} & & & & & \lambda^{(b_3)} & \ge -20 \\
\lambda^{(a_1)} & +\lambda^{(a_2)} & +\lambda^{(a_3)} & -\lambda^{(b_1)} & -\lambda^{(b_2)} & -\lambda^{(b_3)} & \ge -150 \\
 & & \lambda^{(a_3)} & & -\lambda^{(b_2)} & & \ge -70 \\
 & & & & \lambda^{(b_2)} & & \ge -130
\end{array}$$

### Parametric Problem and Evocation of Intentional Degeneration

If the solution of a TP contains less than $m + n - 1$ positive (non-null) values of $x_{ij}$, it is called degenerated. Degeneration in a transportation programme means that the optimum solution can be attained, using a lesser number of transportation routes than in a non-degenerated programme. It can often be advantageous to have minimum transportation routes because of speeding up the programme, better vehicles exploitation, and other reasons. On the other hand, the concentration of transport is limited by the roads' capacity.

There are TP of various types, needed various considerations; the economic interpretation of a degenerated solution has therefore to be differentiated according to individual particularities of the treated problem.



Degeneration in a TP can be initiated intentionally by introducing $\lambda_k = \bar{\theta}^{(k)}$, resp. $\lambda_k = \underline{\theta}^{(k)}$. After having modified the vector $\mathbf{x}_B$ accordingly, we get at least one null-element. Hence, by means a suitable modification of one or more supply – demand values in a TP, we can intentionally evoke degeneration and lessen the number of transportation routes to be used. This can be demonstrated on the example above. Let us suppose $\lambda^{b_2} = -80$, i.c. $b_2 = 50$. We get, in accordance with (6)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 100 \\ 250 \\ 200 \\ 150 \\ 50 \\ 120 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ 20 \\ 230 \\ 150 \\ 50 \end{pmatrix} \begin{array}{l} x_{13} \\ x_{21} \\ x_{23} \\ x_{24} \\ x_{31} \\ x_{32} \end{array}$$

As we can see, the same volume of 550 units can be transported by using a lesser number of transportation routes, i.e. five. At the same time, according to (5), the 4th demand increases by 80 units, i.e. $b_4 = 230$.

### Conclusions

Solution of a problem does not end up by computing the formal optimum. This one has to be further analyzed and consequences of various modifications in the transportation programme considered. The final decision becomes then an adjustment between the optimal ideal solution and the practical possibilities.

Formulation of parametric TP-s of various types and analysis of their solutions represent an important information about properties of transformation systems. Sensibility analysis of optimal solution and consideration of modifications in supply-demand values are of basic use for amendments in transportation programmes; at the same time, the internal degeneration provides the possibility of increasing the programme efficiency.

All the necessary computations can easily be performed in the table processor MS Excel.

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**K využití parametrických dopravních úloh.**

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S rychlým rozvojem počítačových informačních technologií je v poslední době věnována malá pozornost využití parametrických distribučních úloh lineárního programování. Význam těchto úloh spočívá v možnosti lépe poznat vlastnosti zkoumaného systému a využít je pro efektivní rozhodování.

V práci je věnována pozornost parametrickým jednostupňovým dopravním úlohám. Tyto úlohy jsou specifickými úlohami LP, a proto se pro jejich zkoumání využívají speciální přístupy a metody.

Je formulována parametrická dopravní úloha (1) až (5) a odvozeny vztahy (10), (11) a (16), umožňující zjistit interval, resp. oblast přípustných hodnot parametrů při změně okrajových hodnot úlohy v rámci kterých nedochází ke změně báze. Získané poznatky jsou důležité pro analýzu dopravního systému. Zároveň je uveden postup pro vyvolání úmyslné degenerace, která poskytuje z ekonomického hlediska zajímavá řešení. Uvedené vztahy jsou demonstrovány na ilustračních příkladech.

parametrická úloha; lineární parametrizace; matice transformace; interval přípustných hodnot parametrů; oblast přípustných hodnot parametrů; degenerované řešení; stabilita optimálního řešení

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