$\langle u, 1\rangle$ UTILITY FUNCTION IN MULTIPLE CRITERIA DECISION MODELS

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Decision theory and decision models enable the decision-maker to make rational decisions under the more or less known future situation. In practice, the best decision alternative is often chosen under more decision criteria, too. So the decision problem with m alternatives, n states of nature and k criteria can be defined. In this paper an analysis of decision alternatives using the special form of utility function – $\langle u, 1\rangle$ utility function – in the multiple criteria case is described. An example of this function and its application is demonstrated and discussed herein. The advantage of $\langle u, 1\rangle$ utility function lies within the possibility of better analysis of positive or negative payoff values, which is impossible using the von Neumann utility. When utility values are aggregated using a weighted sum of values, risk profiles of all alternatives can be analysed more realistically because the aggregated utility value can be decreased by the negative utility values.

multiple criteria decision table; three-dimensional matrix; utility function; criteria aggregation

INTRODUCTION

Decision-making is a process where one decision alternative has to be chosen from a certain list of real possible decision alternatives. A standard theory of decision-making is based on a theory of game against nature and supposes one criterion only (Bonini et al., 1997; von Neumann, Morgenstern, 1947; Stevenson, 1989; Turban, Meredith, 1991). However, problems of a multiple criteria decision-making are very frequent in everyday life. In such cases there are two or more criteria included in the decision situation. So, the best decision alternative is commonly chosen under more decision criteria (Brožová, 2004a, b; Jones, Mardle, 2004; Kahneman, Tversky, 2000).

Suppose now a situation when one decision alternative has to be chosen from the list of possible alternatives, which are evaluated according to more than one criterion and their results depend on possible future situations. The decision problem with m alternatives, n states of nature and k criteria has to be defined in this case.

There are many approaches to choosing the best choice in multiple criteria models. The aggregation of criteria is one of them. The difficulty is that there is a number of different aggregation procedures. The proper aggregation of criteria payoffs converts the multiple criteria decision problem to a single criterion decision problem. However, this technique includes many well-known disadvantages and can fail especially when compensation among criteria is impossible. Nevertheless, this approach helps to solve problems using single criterion methods.

Many problems with the correctness of aggregation can occur when the criteria evaluation of alternatives have different units. In this case some transformation is necessary. Payoff transformation to one scale is generally based on the utility function (von Neumann, Morgenstern, 1947; Kahneman, Tversky, 2000). Utility values are assigned to all payoffs according to all criteria. Payoffs values are transformed into interval $\langle 0, 1\rangle$, value 0 corresponds to the worth payoff and value 1 to the best payoff. Other payoff values are transformed using the proper utility function. The great advantage of this approach is the possibility of including decision-maker's attitude towards the risk into the problem analysis.

Disadvantage of this form of utility can occur, for instance, when the worst payoff value according to one criterion represents the lowest profit (positive value) and the worst value according to other criterion represents the highest loss (negative value). In this case, utility value 0 has a meaning of the greatest loss and also the lowest profit. Consequently, the opportunity of loss or profit does not have to be so obvious, while analysing utility values.

Therefore a different way of payoffs transformation to the utility will be suggested in this paper.

MATERIAL AND METHODS

For a selection of the best alternatives under the risk, a decision model is typically used in the form of a decision table (a payoff matrix). This model form will be extended to describe the multiple criteria decision problem.

The dominance approach, especially the risk profile analysis, will be used to analyse and solve the problem. In the case of multiple criteria, the transformation of payoffs using special form of utility function called $\langle u, 1\rangle$ utility function and weighted sum of these utility values are suggested.

Decision-making model

The decision model (Bonini et al., 1997; von Neumann, Morgenstern, 1947; Stevenson, 1989; Turban, Meredith, 1991) serves as a methodology for decision making, for choosing the best decision.

Table 1. Single criterion decision table

		States of nature			
		S_1	S_2		S_n
Alternatives	A_1	v_{11}	v_{12}	...	v_{1n}
	A_2	v_{21}	v_{22}	.....	v_{2n}
		.....	.....	.....	.....
	A_m	v_{m1}	v_{m2}	.....	v_{mn}

A_i – the i-th alternative, $i = 1, \ldots, m$,
S_j – the j-th state of nature, $j = 1, \ldots, n$,
v_{ij} – the payoff of alternative A_i / state of nature S_j combination

Its aim is to select one of the decision-maker strategies – alternatives that are available. Their effect depends on possible future states of nature. Payoffs are associated with each combination alternative and state of nature. The appropriate alternative is set by maximising output or minimising input according to decision criterion.

There are two commonly used forms of the decision model – a payoff table or a decision tree. We will use the first one. The general form of a decision table is a two-dimensional matrix (Table 1).

Multiple criteria decision-making model

The best decision alternative is often chosen under more decision criteria. The dimension of the decision matrix has to be extended to represent a multiple criteria decision problem (Brožová, 2004a, b).

The decision problem with m alternatives, n states of nature and k criteria is defined in this case. This decision problem can be formulated in the form of a three-dimensional matrix $W = (w_{ijk})_{m.n.r}$, where w_{ijk} is the payoff of alternative A_i under state of nature S_j and criterion C_k. The first dimension (rows) represents alternatives $A_1, A_2, \ldots, A_m$, the second dimension (columns) represents states of nature $S_1, S_2, \ldots, S_n$ and the third one (sheets in MS Excel terminology) represents criteria $C_1, C_2, \ldots, C_r$ (Fig. 1).

In a typical case, the decision-maker has information for the forecast of probability of the states of nature. These probabilities can be represented as a probability vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$, where probability p_j, $j = 1, \ldots, n$ is probability of the state of nature S_j and $\sum_{j=1}^{n} p_j = 1$.

Probabilistic dominance

There are many risk and decision analysis tools (Bonini et al., 1997; von Neumann, Morgenstern, 1947; Stevenson, 1989; Turban, Meredith, 1991), but most of them focus on one specific analytical technique in decision tables or decision trees. In the decision under the risk with maximisation criterion, it is possible to analyse the probability that alternative payoffs overachieve the required value and to compare these probabilities for different alternatives. Obviously, an alternative with a greater chance of overachieving the required value of payoff is preferred.

Alternative A_i dominates alternative A_j by probabilistic dominance if $P(v_i \geq x) > P(v_j \geq x)$ for all values of x, where x is the required payoff value, v_i is the conditional payoff of i-th alternative and v_j is conditional payoff of j-th alternative.

A main result of this analysis is a distribution of possible outcomes and the probability of getting these results. The cumulative probability $P(w_i \geq x)$ is called a **risk profile** of alternative A_i. Risk profiles of all alternatives can be plotted in one graph. Their analysis can be easily made based on this graphical representation.

Von Neumann's utility

Utility (von Neumann, Morgenstern, 1947; Kahneman, Tversky, 2000) is generally used as a measure of the satisfaction gained by realizing the chosen alternative. By this given measure, a decision-maker may speak meaningfully about increasing or decreasing the utility, and thereby the decision-making can be explained in terms of rational attempts to increase utility. The

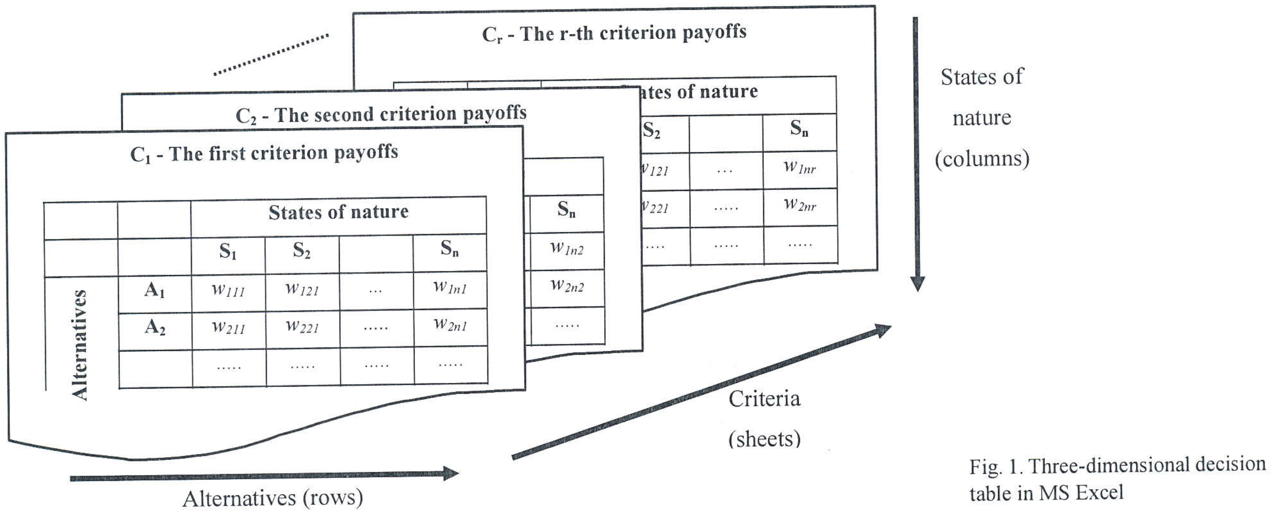

Fig. 1. Three-dimensional decision table in MS Excel

decision-maker's utility function f assigns a utility value to each payoff in the decision table

$$f : w_{ijk} \rightarrow \langle 0, 1 \rangle$$

where value 0 corresponds to the worth payoff and value 1 corresponds to the best payoff. Other payoff values are transformed using proper utility function, describing the decision maker attitude to the risk. If $f(x) > f(y)$, then the decision-maker strictly prefers payoff x to y.

Utility value 0 can have a meaning of the maximum loss according to the one criterion and the minimum profit according to another criterion. So the loss or the profit could seem similar while analysing utility values.

$\langle u, 1 \rangle$ utility function f''

To avoid the above problems we suggest (Brožová, 2004b) another approach to define the utility function. The best payoff value is again transformed to the utility value set as 1. Then the decision-maker identifies payoff value corresponding to the utility value set as 0. This payoff value can be either hypothetical or real. Now, the decision-maker selects the proper form of utility function. So each payoff is transformed into interval $\langle u, 1 \rangle$ and holds following formulas

$$f''(best) = 1$$
$$f''(zero) = 0$$
$$f''(worth) = u$$

where f'' is $\langle u, 1 \rangle$ utility function, *best* is the best payoff value, *zero* is the payoff value with utility equal to 0 and *worth* is the worth payoff value.

Utility value u represents the worth payoff with respect to the utility value equal to 1, which represents the best payoff value. This transformation function can be called $\langle u, 1 \rangle$ **utility function**.

Suppose now, the decision-maker that is indifferent to the risk. This transformation can be expressed using the linear utility function by following formulas

$$f''(best) = 1$$
$$f''(zero) = 0$$
$$f''(value) = \frac{value - zero}{best - zero}$$

where f'' is a linear $\langle u, 1 \rangle$ utility function, *best* is the best payoff value, *zero* is payoff value with utility equal to 0 and *value* is the payoff to be transformed. The utility value u is calculated using following formula

$$u = \frac{worth - zero}{best - zero}$$

where *best* is the best payoff value, *zero* is the payoff value with the utility equal to 0 and *worth* is the worth payoff value.

The alternative analysis in multiple criteria decision model using utility aggregation

The analysis of decision alternatives (Bonini et al., 1997; von Neumann, Morgenstern, 1947; Stevenson, 1989; Turban, Meredith, 1991), which are evaluated according to more than one criterion and their results depend on possible future situations, can be easily made by using proper utility function because transformed payoff values according to all criteria have the same unit.

When the $\langle u, 1 \rangle$ utility function is used, the utility value u is generally different for different criteria. So, now the transformed payoffs can be compared and also the alternatives can be analysed with respect to evidently bad payoff values.

The best tool to provide the alternatives analysis is the risk profile. In the case of multiple criteria decision-making under the risk, each alternative can be represented by r risk profiles, which can be plotted in the one graph and compared with the other alternatives through the same representation. The biggest advantage of this approach is the possibility to analyse all data without great biases. It enables a global view on the utility of each alternative. The biggest disadvantage is a confusion when $r * m$ risk profiles are plotted together to one graph.

Consequently, we suggest using aggregation of utility values by the weighted sum method. The utility values and information of criteria preferences articulated by the decision maker are necessary information for this calculation. The following formula will be used

$$F''_{ij} = \sum_{k=1}^{r} \omega_k f''_k(w_{ijk})$$

where F''_{ij} is the aggregated utility value for alternative A_i under the state of nature S_j, ω_{ijk} is the preference of criterion C_k, f''_k is $\langle u, 1 \rangle$ utility function according to criterion k and w_k is the payoff of alternative A_i under the state of nature S_j and criterion C_k.

In this case, the risk profile of each alternative is provided using aggregated utility values and the simple comparison of as many risk profiles as alternatives is performed.

RESULTS

Application of the approach described above will be shown in the following simple decision problem. We have to analyse three alternatives A1, A2 and A3, whose payoffs according to three criteria MAX1, MAX2 and MIN depend on four possible future states of nature S1, S2, S3 and S4. This problem can be represented as a three-dimensional decision matrix (Table 2).

First of all we use the linear $\langle u, 1 \rangle$ utility function for payoffs transformation. Payoffs value *best* and *zero* are set for all three criteria (Table 3) and $\langle u, 1 \rangle$ utility of payoffs is calculated.

Table 2. Three-dimensional decision matrix

MAX 1	S1	S2	S3	S4	Preference
A1	–5	10	12	3	0.5
A2	6	4	9	–2	
A3	1	2	–1	6	
MAX 2	**S1**	**S2**	**S3**	**S4**	
A1	2	10	7	3	0.35
A2	6	4	2	1	
A3	1	2	4	6	
MIN	**S1**	**S2**	**S3**	**S4**	
A1	4	9	7	2	0.15
A2	8	4	3	3	
A3	5	3	0	2	
Probability	0.1	0.3	0.2	0.4	

Table 3. Decision matrix with transformed payoffs using linear u-1 utility function

MAX 1	S1	S2	S3	S4	*Best*	*Zero*
A1	–0.41667	0.833333	1	0.25	12	0
A2	0.5	0.333333	0.75	–0.16667		
A3	0.083333	0.166667	–0.08333	0.5		
MAX 2	**S1**	**S2**	**S3**	**S4**	***Best***	***Zero***
A1	0.111111	1	0.666667	0.222222	10	1
A2	0.555556	0.333333	0.111111	0		
A3	0	0.111111	0.333333	0.555556		
MIN	**S1**	**S2**	**S3**	**S4**	***Best***	***Zero***
A1	0.428571	–0.28571	0	0.714286	0	7
A2	–0.14286	0.428571	0.571429	0.571429		
A3	0.285714	0.571429	1	0.714286		
Probability	0.1	0.3	0.2	0.4		

Payoffs for criterion MAX1 are transformed using formula $f''_{MAX1}(value) = \dfrac{value - 0}{12 - 0}$.

Payoffs for criterion MAX2 are transformed using formula $f''_{MAX2}(value) = \dfrac{value - 1}{10 - 1}$.

Payoffs for criterion MIN are transformed using formula $f''_{MIN}(value) = \dfrac{value - 7}{0 - 7}$.

The following decision tables with u-1 utility values are obtained (Table 3).

The second step consists of utility aggregation. Aggregated values are calculated as a weighted sum of partial utility values with preference of criterion MAX1 equal to 0.5, preference of criterion MAX2 equal to 0.35 and preference of criterion MIN equal to 0.15 (Table 4). The formula of aggregation is

$$F''_{ij} = \omega_1 f''_1(w_{ij1}) + \omega_2 f''_2(w_{ij2}) + \omega_3 f''_3(w_{ij3}) =$$

$$= 0.5 f''_1(w_{ij1}) + 0.35 f''_2(w_{ij2}) + 0.15 f''_3(w_{ij3})$$

The third step is the construction of risk profiles of all three decision-alternatives. Cumulative probabilities representing a distribution of possible utility values of all three alternatives are in Table 5.

Three risk profiles are plotted in Fig. 2 and their analysis can be performed.

Table 4. Aggregated decision matrix with u-1 utility function

MAX	S1	S2	S3	S4
A1	–0.105	0.7238	0.7333	0.3099
A2	0.423	0.3476	0.4996	0.0024
A3	0.0845	0.2079	0.225	0.5516
Probability	0.1	0.3	0.2	0.4

The last step is the most important one, because it includes risk profile analysis. This analysis shows that there is no strictly dominating alternative according to the preferences of criteria. The worth utility value of alternatives is negative in case of alternative A1, is zero in case of alternative A2 and is positive in case of alternative A3.

Risk preferring decision-makers can choose alternative A1, because it has greater probability of higher utility values. Alternatives A2 and A3 are better in case of the worth values of utility.

Decision-makers who are indifferent to risk may also prefer the alternative A1, because it yields relatively good payoffs with relatively high probability.

Risk averse decision-makers would have to choose alternative A3, which is better in area of lower values of utility and which is better then alternative A2 but its best result is not good enough in comparison with alternative A1.

Table 5. Cumulative probability of aggregated utility

A1	–0.105	–0.105	0.3099	0.3099	0.7238	0.7238	0.7333	0.7333
Cumulative probability	1	0.9	0.9	0.5	0.5	0.2	0.2	0.00
A2	0.0024	0.0024	0.3476	0.3476	0.423	0.423	0.4996	0.4996
Cumulative probability	1	0.6	0.6	0.3	0.3	0.2	0.2	0.00
A3	0.0845	0.0845	0.2079	0.2079	0.225	0.225	0.5516	0.5516
Cumulative probability	1	0.9	0.9	0.6	0.6	0.4	0.4	0.00

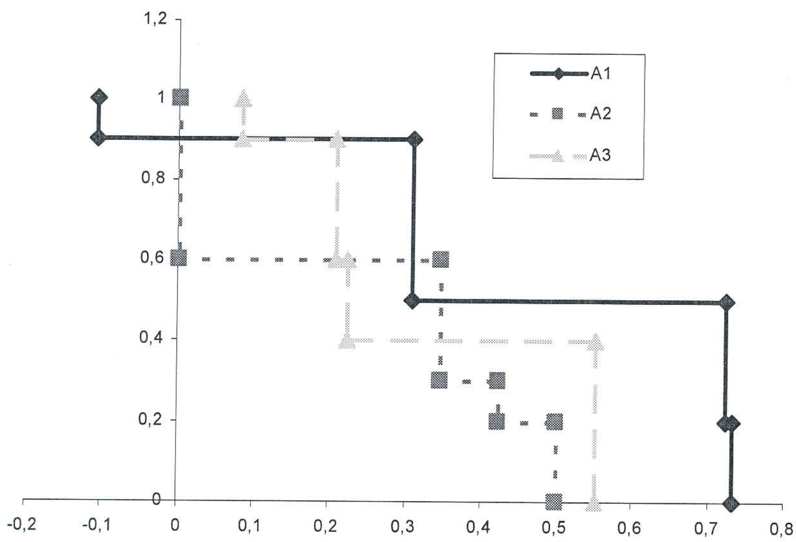

Fig. 2. Risk profile for multiple criteria decision matrix

We can conclude that the most sufficient alternative is A1. Alternatives A3 and A2 can be chosen only in the case of extremely pessimistic decision-maker.

DISCUSSION

The approach to the risk profile construction and analysis in multiple criteria decision table using $\langle u, 1\rangle$ utility function has been described in this paper.

– It is based on payoffs transformation for each combination alternative/state of nature using $\langle u, 1\rangle$ utility function, e.g. with respect to the best value (it is transformed to 1) and to the indifferent value (it is transformed to 0). The worst payoff value is transformed to value u, which can be negative.
– This $\langle u, 1\rangle$ utility function enables the analysis of positive or negative payoff values, which is impossible using the von Neumann utility.
– Utility values can be aggregated using the weighted sum method, so the aggregated utility value is decreased by the negative utility values.
– Risk profiles of all alternatives can be easily graphically analysed. This graph enables a global view on the utility of each alternative.

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$\langle u, 1\rangle$ funkce užitku ve vícekriteriálních rozhodovacích modelech.

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Rozhodovací modely poskytují rozhodovateli kvantitativní podporu racionálního rozhodování. Tyto modely v klasické formě umožňují vybrat nejvhodnější alternativu rozhodnutí, její výsledný efekt je ovlivňován budoucím vývojem, budoucím stavem světa. Přitom se předpokládá, že výsledný efekt je stanoven z jediného hlediska, kritéria. Ve skutečnosti je však většina rozhodovacích procesů vícekriteriálních, to znamená, že většina rozhodnutí je vybírána podle více kritérií.

V postupech řešení takovýchto problémů se velmi často užívá hodnocení jednotlivých rozhodnutí podle užitku, který poskytují. Klasická von Neumannova funkce užitku vychází z principu, že nejlepší kriteriální hodnocení poskytuje jednotkový užitek a nejhorší výsledek odpovídá nulovému užitku. Výhodou v případě vícekriteriálních rozhodovacích situací je, že funkce užitku umožňuje agregovat jednotlivé hodnoty užitku a tak vlastně převést vícekriteriální problém na jednokriteriální. Nevýhodou von Neumannovy funkce užitku je možné setření rozdílů mezi špatnými ohodnoceními jednotlivých rozhodnutí. Nejmenší zisk (kladný výsledek) podle jednoho kritéria totiž bude představován stejným nulovým užitkem jako největší ztráta (záporný výsledek) podle jiného kritéria.

Cílem tohoto příspěvku je ukázat speciální formu funkce užitku nazvanou ⟨*u*, 1⟩ funkce užitku, vysvětlit její konstrukci a ukázat možnosti jejího využití v analýze profilu rizika jednotlivých alternativ rozhodnutí při vícekriteriálním rozhodování za rizika.

Vícekriteriální rozhodovací model obsahuje *m* alternativ, *n* stavů okolností a *k* kritérií a $m*n*k$ výplat pro každou možnou kombinaci (alternativa/stav okolností/kritérium). Tento model je tedy zobrazen tolika rozhodovacími tabulkami, kolik kritérií je při rozhodnutí bráno v úvahu. Soubor těchto rozhodovacích tabulek je možno chápat jako třídimenzionální tabulku, ve které první dimenze (řádky) je dimenzí alternativ, druhá dimenze (sloupce) odpovídá stavům okolností a třetí dimenze (listy) představuje jednotlivá kritéria. Nejčastěji jsou tyto modely používány pro výběr nejlepší alternativy při rozhodování za rizika, to znamená, že také musí být známé nebo odhadnuté pravděpodobnosti jednotlivých stavů okolností.

Velmi silným nástrojem pro řešení rozhodovacích situací za rizika je analýza profilu rizika jednotlivých alternativ, která je rozšířením a zobecněním principu dominance podle pravděpodobností. Na základě této analýzy je vybrána alternativa s nejvyšší pravděpodobností výplaty požadované velikosti a vyšší.

Aby bylo možno analyzovat jednotlivé alternativy rozhodnutí pomocí nástrojů jednokriteriálního rozhodování, je použita transformace modelu pomocí agregace hodnot ⟨*u*, 1⟩ funkce užitku. Tato funkce stejně jako klasická von Neumannova funkce užitku předpokládá, že jednotkový užitek poskytuje nejlepší ohodnocení. Dále musí být stanovena (i hypotetická neexistující) hodnota kritéria, která odpovídá nulovému užitku. Nulový užitek tedy nemusí odpovídat nejhoršímu ohodnocení. Hodnota užitku ostatních kriteriálních hodnot pak odpovídá zvolené funkci užitku (lineární, nelineární apod.). To znamená, že hodnota užitku nejhoršího ohodnocení je obecně rovna nenulové (záporné nebo i kladné) hodnotě *u*. Všechny hodnoty užitku potom leží v intervalu ⟨*u*, 1⟩.

Postup řešení ilustračního vícekriteriálního rozhodovacího modelu, definice lineární ⟨*u*, 1⟩ funkce užitku, agregace užitkových hodnot a analýza profilu rizika jednotlivých alternativ ukazují základní výhody tohoto způsobu výběru nejlepší alternativy:

- Tato funkce užitku umožňuje rozlišit i míru nevýhodnosti jednotlivých kriteriálních hodnot. I malý zisk může představovat určitý kladný užitek a ztráta by měla být ohodnocena negativním užitkem.
- To se pochopitelně projeví i při agregaci užitkových hodnot v případě vícekriteriálních rozhodovacích modelů.
- Tak je možno vícekriteriální rozhodovací model transformovat na jednokriteriální s menším zkreslením a použít analýzu profilu rizika pro volbu nejvýhodnější alternativy.

vícekriteriální rozhodovací tabulka; třídimenzionální matice; funkce užitku; agregace kritérií

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